A Survey for exotic 4-manifolds

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Introduction

This note is a survey of the papers I have read and techniques I have learned about the constructions for exotic pairs.

I am only barely getting started on this survey. I will (try to) keep updating this survey biweekly on my website yikaiteng.net. Please let me know if you have any comments/find any typos.

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Exotica from Elliptic Fibrations

3.1 Exotic Elliptic Surfaces

In this section we will apply the techniques introduced in Chapter 2 and construct an infinite family of exotic elliptic surfaces $E(n)$.

3.2 Elliptic Surfaces as Double Branched Covers

In this section we survey the results from Stipsicz and Szabo's result in [SS23], and study a family of exotic definite 4-manifolds.

To be specific,

Chapter 4

Exotic Small $m\mathbb{CP}^2\#n\overline{\mathbb{CP}^2}$'s

4.1 Exotic $\mathbb{CP}^2\#3\overline{\mathbb{CP}^2}$ and exotic $3\mathbb{CP}^2\#5\overline{\mathbb{CP}^2}$

In this section we follow [AP08] and construct exotic $\mathbb{CP}^2\#3\overline{\mathbb{CP}^2}$ and exotic $3\mathbb{CP}^2\#5\overline{\mathbb{CP}^2}$.

4.1.1 Preliminaries

4.1.2 Constructions

We start with the trefoil knot $K := 3_1$. Denote its 0-surgery by $M_k := S^3_0(K)$. Since we know that the trefoil is a genus 1 fibered knot. Thus M_k can be seen as a $\mathbb{T}^2$ bundle over S^1. We call one of its sections b.

By taking a product with S^1, $M_k \times S^1$ is a $\mathbb{T}^2$ bundle over $\mathbb{T}^2$. We denote F by one of its fiber, and S as the section $b \times S^1$. By a result of Thurston [Thu22], we know that $M_k \times S^1$ yields a symplectic structure such that both F and S are its symplectic submanifolds.

Now we take the symplectic sum of $C_S := (M_k \times S^1)\backslash\nu(S)$ and $C_F := (M_k \times S^1)\backslash\nu(F)$, and we call the resulting closed 4-manifold $Y_K := C_S\#_\psi C_F$. Here ψ is a boundary diffeomorphism between C_F and C_S, whose action on the fundamental groups will be specified in the next subsection.

In the manifold Y_K, we can locate a submanifold Σ_2, obtained by taking the connected sum of a section of C_F and a fiber of C_S. Moreover, this submanifold is a genus 2 surface, and can be made a symplectic submanifold.

Now we take the symplectic sum of two copies of Y_K along the submanifold Σ_2. This gives the manifold $X_K := Y_K\#_{\Sigma_2}Y_K$. Again the boundary diffeomorphism is specified in the next section. In fact, this is the cohomology $S^2 \times S^2$ constructed in [Akh08]. The two manifolds X_K and Y_K are the building blocks of the constructions of exotic $\mathbb{CP}^2\#3\overline{\mathbb{CP}^2}$ and $3\mathbb{CP}^2\#5\overline{\mathbb{CP}^2}$.

We first construct an exotic $3\mathbb{CP}^2\#5\overline{\mathbb{CP}^2}$, denoted X. We start with the 4-torus $\mathbb{T}^4 = \mathbb{T}^2 \times \mathbb{T}^2$ with the symplectic structure obtained by pulling back two copies of the standard symplectic structure on $\mathbb{T}^2$, which in formula writes as $\omega := p_{1,2}^*(\omega_{1,2}) + p_{3,4}^*(\omega_{3,4})$. In this symplectic $\mathbb{T}^4$, we can locate a genus 2 surface Σ_2' by taking the connected sum of two copies of $\mathbb{T}^2$. Note that Σ_2' has self-intersection number 2 in $\mathbb{T}^4$, and thus have self-intersection number 0 in the blown-up-twice manifold $\mathbb{T}^4\#2\overline{\mathbb{CP}^2}$. The closed 4-manifold X is then defined as the symplectic sum of X_K and $\mathbb{T}^4\#2\overline{\mathbb{CP}^2}$, gluing Σ_2 and Σ_2' together. Again the boundary diffeomorphism is specified in the next section. We claim that this X is an exotic $3\mathbb{CP}^2\#5\overline{\mathbb{CP}^2}$.

Finally we construct an exotic $\mathbb{CP}^2\#3\overline{\mathbb{CP}^2}$, denoted U. Recall that in $M_K \times S^1$, we can locate a genus 2 surface by taking the connected sum of a torus fiber F and a torus section S. Again this submanifold has self-intersection number 2 in $M_K \times S^1$ and thus has self-intersection number 0 in the blown-up-twice manifold $M_K \times S^1\#2\overline{\mathbb{CP}^2}$. Now we define U by taking the symplectic sum of Y_K and $M_K \times S^1\#2\overline{\mathbb{CP}^2}$, and claim that it is an exotic copy of $\mathbb{CP}^2\#3\overline{\mathbb{CP}^2}$.

4.1.3 Calculations of Fundamental Groups

In this subsection, we calculate the fundamental groups of each 3- and 4-manifold constructed in the previous subsection. In the meantime, we use the fundamental group data to specify several boundary diffeomorphisms when doing symplectic sums above.

Recall that the fundamental group for M_K is the knot group for the trefoil $\pi_1(M_K) \cong \langle a, b \,|\, aba = bab \rangle$.

Lemma 4.1.

4.1.4 Proving Exotica

4.1.5 Kirby Diagrams

For this subsection, we follow the routine of

Part III

Misc

Chapter 5

Exotic $\mathbb{R}^4$'s

5.1 By plug twisting 0-traces

5.2 Using Casson handles

a

5.3 Relations between the constructions

a

Chapter 6

Misc

6.1 LLP23

In this section we follow [LLP23] and introduce a Heegaard theoretic approach to detect closed exotic 4-manifolds.

6.1.1 Invariants for 4-manifolds with Positive b_1

In this section, we define an invariant for smooth 4-manifolds with positive b_1, based on Heegaard Floer Theory.

Definition 1 (The α-invariant)**.** Suppose X is a closed oriented connected smooth 4-manifold with $b_1 > 0$, and $\eta \in H_3(X)$ a primitive element (that is, not a positive integer multiple of another element in the lattice). Then we define the α invariant $\alpha(X, \eta)$ to be the minimal $\mathbb{F}$-dimension of $HF_{red}(Y)$, where Y is a smoothly embedded closed connected oriented 3-manifold representing the homology class η.

Note that $\alpha(X, \eta) = \alpha(X, -\eta)$, as the dimension for the reduced Heegaard Floer homology does not distinguish a 3-manifold from its orientation inverse. Thus if $b_1(X) = 1$, we simply write $\alpha(X) := \alpha(X, \eta)$, as there is only one generator for $H_3(X)$ up to a change of orientation.

Example. For a 4-manifold X of the form $S^1 \times S^3 \# Z$, where Z is a closed oriented smooth 4-manifold with $b_1(Z) = 0$, its α-invariant $\alpha(X) = 0$, as the S^3 component represents the generator for $H_3(X)$.

In general, the α-invariant is hard to calculate. However, for a specific case as described by the following proposition, we know exactly how to calculate the α-invariant.

Proposition 6.1 (Proposition 4.3 of [LLP23])**.** *Let Y be a closed connected oriented 3-manifold, and W a cobordism from Y to itself such that the induced map on reduced*

Heegaard Floer homology (summing over all spin^c structures)

$$F_W : HF_{red}(Y) \to HF_{red}(Y)$$

is an isomorphism. Gluing the two ends of W yields a closed 4-manifold X, in which Y represents a third homology class which we call η. Then the submanifold Y actually minimizes the dimension of HF_{red} for the class η, i.e. $\alpha(X, \eta) = dim\ HF_{red}(Y)$. Moreover, if $HF_{red}(Y) \neq 0$, then X does not contain any embedded 2-spheres with self intersection ± 1.

Proof. □

Applying the above proposition to a settings where the homeomorphism type is completely classified , e.g. $\pi_1(X) \cong \mathbb{Z}$, we can modify the proposition for the purpose of generating exotic pairs, as described by the following proposition.

Proposition 6.2. *Suppose Y is a homology 3-sphere, W a cobordism between Y and itself with odd intersection form, and X the closed 4-manifold obtained by gluing the two ends of W via some diffeomorphism. Then X is homeomorphic to $S^1 \times S^3 \# m\mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}$, where $m = b_2^+(W)$ and $n = b_2^-(W)$. Then, if $HF_{red}(Y) \neq 0$ and W induces an isomorphism on $HF_{red}(Y)$, then X is an exotic copy of $S^1 \times S^3 \# m\mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}$.*

6.1.2 Invariants for Simply-connected 4-manifolds

6.1.3 Building Blocks

6.1.4

6.1.5

6.1.6

Bibliography

[Akh08] Anar Akhmedov. Construction of symplectic cohomology $S^2 \times S^2$. In *Proceedings of Gökova Geometry-Topology Conference 2007*, pages 36–48. Gökova Geometry/Topology Conference (GGT), Gökova, 2008.

[AP08] Anar Akhmedov and B. Doug Park. Exotic smooth structures on small 4-manifolds. *Invent. Math.*, 173(1):209–223, 2008.

[LLP23] Adam Simon Levine, Tye Lidman, and Lisa Piccirillo. New constructions and invariants of closed exotic 4-manifolds, 2023.

[SS23] Andras I. Stipsicz and Zoltan Szabo. Exotic definite four-manifolds with non-trivial fundamental group, 2023.

[Thu22] W. P. Thurston. Some simple examples of symplectic manifolds. In *Collected works of William P. Thurston with commentary. Vol. I. Foliations, surfaces and differential geometry*, pages 617–618. Amer. Math. Soc., Providence, RI, [2022] ©2022. Reprint of [0402764].